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$sl_q(2)$ realizations for Kepler and oscillator potentials and q-canonical transformations

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Abstract. The realizations of the Lie algebra corresponding to the dynamical symmetry group $SO(2, 1)$ of the Kepler and oscillator potentials are q-deformed. The q-canonical transformation connecting two realizations is given and a general definition for the q-canonical transformation is deduced. A q-Schrödinger equation for a Kepler-like potential is obtained from the q-oscillator Schrödinger equation. The energy spectrum and the ground-state wavefunction are calculated.

1. Introduction

There are mathematical and physical aspects of q-deformations [1]. From the mathematical point of view, one usually demands that the q-deformed algebra be a Hopf algebra. The physical point of view is somehow less restrictive: obtaining the underlying undeformed picture in the $q \to 1$ limit is the basic condition. Hence, q-deformation of a physical system is not unique. For example, the harmonic oscillator which is the most extensively studied system has several q-deformed descriptions [2]. q-deformation of physical systems other than the oscillator are not well studied as most of the concepts of classical and quantum mechanics become obscure after q-deformations. For example, q-deformed change of phase-space variables leaving basic q-commutation relations invariant is presented in [3], and a canonical transformation connecting q-oscillators is studied in [4]; but q-canonical transformations establishing relationships between different potentials are not known.

The purpose of this paper is to present a q-canonical transformation and to define a q-deformed Kepler-like potential in a consistent manner with the q-oscillator. The possession of the same dynamical symmetry group $SO(2, 1)$ by the harmonic oscillator and the Kepler problems will guide us.

In general, the phase-space realizations of the Lie algebras corresponding to a dynamical symmetry group which are relevant to different physical systems are connected by canonical transformations. We generalize this connection to define q-canonical transformations. We hope that the procedure may also help to define new q-deformed potentials from the known ones.

In section 2 we review the known relation between the undeformed Kepler and oscillator problems. In section 3 we present q-deformations of two realizations of $sl(2)$ (which is the Lie algebra of $SO(2, 1)$) relevant to the Kepler and oscillator potentials. We define the

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q-canonical transformation connecting two realizations. We then give a general definition of q-canonical transformation. In section 4 we define a q-Schrödinger equation for a Kepler-like potential from the q-Schrödinger equation of the q-oscillator by a coordinate change. Finally, we fix the energy spectrum of the q-oscillator, and find the ground-state wavefunction and then obtain the energy spectrum and the ground-state wavefunction of the q-deformed Kepler problem.

2. Review of the relations between Kepler and oscillator potentials

It is well known that $SO(2,1)$ is the dynamical symmetry group of the radial parts of the Schrödinger equations of the Kepler and the harmonic oscillator potentials†.

In one (space) dimension the phase-space realizations of the corresponding Lie algebra $sl(2)$ relevant to the Kepler and the harmonic oscillator problems are given by

$$\begin{aligned} H &= 2px \\ X_+ &= -\sqrt{2}\,x \\ X_- &= \frac{-1}{\sqrt{2}}\,p^2x \end{aligned} \tag{1}$$

and

$$\begin{aligned} L_0 &= up_u + \frac{\mathrm{i}}{2} \\ L_+ &= -\sqrt{2}\,u^2 \\ L_- &= -\frac{1}{4\sqrt{2}}\,p_u^2 \end{aligned} \tag{2}$$

with

$$px - xp = \mathrm{i} \qquad p_u u - up_u = \mathrm{i}.$$

The above generators satisfy the usual commutation relations

$$[H, X_\pm] = \pm 2\mathrm{i}X_\pm \qquad [X_+, X_-] = -\mathrm{i}H \tag{3}$$

$$[L_0, L_\pm] = \pm 2\mathrm{i}L_\pm \qquad [L_+, L_-] = -\mathrm{i}L_0. \tag{4}$$

The eigenvalue equation for the Kepler Hamiltonian

$$H_{\mathrm{K}}\Psi \equiv \left(\frac{p^2}{2\mu} + \frac{\beta^2}{x}\right)\Psi = E\Psi$$

which is equivalent to

$$\left(\frac{p^2}{2\mu} - E\right)x\Phi = \beta^2\Phi$$

is solved by diagonalizing the operator

$$\frac{1}{\sqrt{2}}\left(\frac{1}{\mu}X_- + EX_+\right).$$

On the other hand, the solution of the oscillator problem is simply obtained by diagonalizing the operator

$$-\frac{1}{\sqrt{2}}\left(\frac{1}{\mu}L_- + \tfrac{1}{2}\mu\omega^2 L_+\right).$$

† See for example [5].

Classically (i.e. before the $\hbar$-deformation) the Kepler and oscillator phase-space variables are connected by the canonical transformation

$$x = u^2 \qquad p = \frac{p_u}{2u}. \tag{5}$$

This type of canonical transformations is also employed for solving the H-atom path integral [6]. In fact, since the path integrations make use of the classical dynamical variables, the canonical point transformations are routinely used to transform the path integral of a given potential into a solvable form.

The relation between the Schrödinger equations corresponding to the one-dimensional oscillator and Kepler type potentials is as follows.

The Schrödinger equation of the one-dimensional oscillator

$$\left(-\frac{1}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \tfrac{1}{2}\mu\omega^2 u^2\right)\psi = E\psi \tag{6}$$

is transformed by the coordinate change suggested by (5)

$$u = \sqrt{x} \tag{7}$$

into

$$\left(-\frac{1}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{E/8}{x} - \frac{3/32\mu}{x^2}\right)\phi = -\frac{\mu\omega^2}{8}\phi \tag{8}$$

with

$$\psi = \frac{1}{\sqrt{x}}\phi. \tag{9}$$

The energy E and the frequency ω^2 of the oscillator play the role of the coupling constant β^2 and the energy E_{K} of the Kepler problem:

$$\frac{E}{8} = \frac{\omega}{4}(2n+1) = -\beta^2 \tag{10}$$

$$E_{\mathrm{K}} = -\mu\omega^2/8. \tag{11}$$

Equation (8) is equivalent to the one-dimensional Kepler problem with an extra potential barrier $-(3/32\mu)/x^2$ or to the two-dimensional Kepler problem with 'angular momentum' $p_\theta^2 = -3/16$.

If we obtain ω from (10) as

$$\omega(\beta) = -\frac{4\beta^2}{2n+1} \tag{12}$$

and substitute it into (11), we obtain the Kepler energy

$$E_{\mathrm{K}} = -\frac{2\mu\beta^4}{(2n+1)^2}. \tag{13}$$

3. q-canonical transformation between the Kepler and oscillator realizations of $sl_q(2)$

3.1. q-deformation of the Kepler realization

To q-deform the algebra of the generators (1), we prefer to q-deform the commutation relation between p and x (we use the same notation for the q-deformed and undeformed objects):

$$xp - qpx = -\mathrm{i}\sqrt{q} \tag{14}$$

but keep the functional forms of H and $X_\pm$ the same as the forms given in (1). The q-deformed commutation relations are then given by [7]

$$\begin{aligned} &HX_- - qX_-H = -2\mathrm{i}\sqrt{q}\,X_- \\ &HX_+ - \frac{1}{q}X_+H = 2\mathrm{i}\frac{1}{\sqrt{q}}\,X_+ \\ &X_+X_- - q^2X_-X_+ = \frac{-\mathrm{i}}{2}\sqrt{q}\,(1+q)H. \end{aligned} \tag{15}$$

Note that after rescaling the above generators:

$$X_\pm \to \frac{\mathrm{i}\sqrt{\sqrt{q}(1+q)}}{\sqrt{q}}X_\pm \qquad H \to 2\mathrm{i}H$$

and setting $q = r^2$ one arrives at

$$\begin{aligned} &HX_- - r^2X_-H = -rX_- \\ &r^2HX_+ - X_+H = rX_+ \\ &X_+X_- - r^4X_-X_+ = r^2H \end{aligned} \tag{16}$$

which is Witten's second deformation of $sl(2)$ [8].

3.2. q-deformation of the oscillator realization

We would like to q-deform the generators given in (2), which are relevant to the oscillator problem, in a consistent manner with the deformation of the Kepler realization (1).

We define the q-deformed commutation relation of p_u and u as

$$up_u - \sqrt{q}\,p_uu = \mathrm{i}b(q) \tag{17}$$

and fix $b(q)$ by requiring that the commutation relations of the q-deformed algebra of the generators (2) to be the same as (15). We rescale L_0 and $L_\pm$:

$$L_0 \to \frac{a}{2\sqrt{q}}L_0 \qquad L_\pm \to \left[\frac{a(1+\sqrt{q})^3}{8\sqrt{q}\,(1+q)}\right]^{1/2} L_\pm$$

with

$$a = \frac{(1+\sqrt{q})(1+q)}{2\sqrt{q}}$$

and fix $b(q)$ as

$$b = -\tfrac{1}{2}\left(1 + \frac{1}{\sqrt{q}}\right).$$

The q-deformed algebra then becomes

$$\begin{aligned} &L_0L_- - qL_-L_0 = -2\mathrm{i}\sqrt{q}\,L_- \\ &L_0L_+ - \frac{1}{q}L_+L_0 = 2\mathrm{i}\frac{1}{\sqrt{q}}L_+ \\ &L_+L_- - q^2L_-L_+ = \frac{-\mathrm{i}}{2}\sqrt{q}\,(1+q)L_0 \end{aligned} \tag{18}$$

which is the same as the $sl_q(2)$ algebra of the Kepler problem (15).

Note that before q-deformation, $sl(2)$ algebra (4) admits three different choices for L_0:

$$L_0 = up_u + \frac{\mathrm{i}}{2} \qquad L_0 = p_uu - \frac{\mathrm{i}}{2} \qquad L_0 = \frac{1}{2}(up_u + p_uu). \tag{19}$$

In the q-deformed case, however, if we require the generators to be independent of q (except an overall factor), the ordering degeneracy in (19) is removed, that is, L_0 can only take the form given in (2).

3.3. *q-canonical transformation*

Let us introduce a transformation similar to (5):

$$x = \left(\frac{u}{b}\right)^2 = \left(\frac{2\sqrt{q}}{1+\sqrt{q}}\right)^2 u^2 \qquad p = \tfrac{1}{2}u^{-1}p_u. \tag{20}$$

Then, the q-commutation relation (14) yields

$$\frac{q}{1+\sqrt{q}}(up_u - qu^{-1}p_u u^2) = -\mathrm{i}\sqrt{q} \tag{21}$$

which is consistent with (17). Indeed, by virtue of (17) the above commutation relation becomes

$$\frac{q}{1+\sqrt{q}}(up_u - \sqrt{q}\,p_u u + \mathrm{i}b\sqrt{q}) = -\mathrm{i}\sqrt{q}$$

which is again equal to (17).

Now, we are ready to define the q-canonical transformation.

Definition. We wish to keep the phase-space realizations of the q-deformed generators to be formally the same as the undeformed generators of the dynamical symmetry group. We then define the transformation $x, p \to u, p_u$ to be the q-deformed canonical transformation if the following two conditions are satisfied.

(i) Algebras generated by the realizations $X_i(x, p)$ and $L_i(u, p_u)$ are the same.

(ii) The q-commutation relations between p and x, and p_u and u are preserved.

In accordance with the above definition, we conclude that (20) is a q-canonical transformation.

By rescaling the q-canonical variables p, x and p_u, u as

$$(x, p) \to q^{-1/4}(x, p) \qquad (u, p_u) \to \sqrt{|b|/\sqrt{q}}\,(u, p_u)$$

and setting

$$q \to q^{-2}$$

the q-commutators (14) and (21) become

$$px - q^2xp = \mathrm{i} \tag{22}$$

$$p_u u - qup_u = \mathrm{i}. \tag{23}$$

In the rest of this paper these q-commutation relations will be used.

There is another definition of the q-deformed canonical transformation [4]: phase-space coordinates are transformed under the condition that the q-commutators remain invariant. However, our condition in the definition of the q-canonical transformation is to obtain, in a suitable limit, the undeformed mappings connecting different potentials which possess the same dynamical symmetry. Thus, our definition of the q-canonical transformation is dynamics-dependent, i.e. the basic q-commutators are potential-dependent (22) and (23).

4. q-canonical transformation from q-oscillator Schrödinger equation to q-Kepler problem

Introduce the q-deformed derivative $D_q(u)$ [9]:

$$D_q(u)f(u) \equiv \frac{f(u)-f(qu)}{(1-q)u}. \tag{24}$$

In terms of this definition one can show that

$$D_q(u)\{f(u)g(u)\} = D_q(u)f(u)g(u) + f(qu)D_q(u)g(u). \tag{25}$$

Since the q-deformed derivative $D_q(u)$ satisfies

$$D_q(u)u - quD_q(u) = 1$$

we can set

$$p_u = \mathrm{i}D_q(u)$$

which is consistent with (23).

In terms of this q-differential realization one can obtain the q-deformed Schrödinger equation for the q-oscillator:

$$\left(-\frac{1}{2\mu}D_q^2(u) + \frac{\mu}{2}\omega_q^2 c_q^2(u) - \frac{1}{2}E_q\right)\psi_q(u) = 0 \tag{26}$$

where

$$c_q(u) = \sqrt{q}\,u \qquad \omega_q = [\omega]_q \equiv \frac{1-q^\omega}{1-q}. \tag{27}$$

Obviously, the choice (27) is not unique†. The conditions to be satisfied are

$$\lim_{q\to 1} c_q(u) = u \qquad \lim_{q\to 1}\omega_q = \omega.$$

We adopt the change of variable suggested by (20)

$$u = \sqrt{x}. \tag{28}$$

The q-derivative $D_q(u)$ transforms as

$$D_q(u) = (1+q)\sqrt{x}\,D_{q^2}(x). \tag{29}$$

$D_{q^2}(x)$ satisfies

$$D_{q^2}(x)x - q^2 x D_{q^2}(x) = 1 \tag{30}$$

hence in accordance with (22) it can be identified with $-\mathrm{i}p$. Therefore, (28) and (29) are equivalent to the q-canonical transformation (20). The q-Schrödinger equation (26) then becomes

$$\left[-\frac{1}{2\mu}(1+q)^2 x D_{q^2}^2(x) - \frac{1}{2\mu}(1+q)D_{q^2}(x) + \frac{1}{2\mu}[\omega]_q^2 qx - \frac{1}{2}E_q\right]\phi_q(x) = 0 \tag{31}$$

with

$$\phi_q(x) = \psi_q(\sqrt{x}). \tag{32}$$

To remove the term linear in $D_q(x)$ in (31) we introduce the ansatz

$$\phi_q(x) = x^\alpha \varphi_q(x). \tag{33}$$

† For an example see [10] and the references given therein.

Choosing

$$\alpha = \frac{\ln((3-q)/2)}{2\ln q}$$

and by multiplying (31) from the left by $1/(1+q)^2 x$, we obtain

$$\left[\frac{-1}{2\mu}D_{q^2}^2(x) - \frac{(2q^2-2q-3)/8\mu q^2(1+q)^2}{x^2} - \frac{E_q/2(1+q)^2}{x}\right]\varphi_q(x) = \frac{\mu[\dot{\omega}]_q^2 q}{2(1+q)^2}\varphi_q(x) \tag{34}$$

which is the q-deformed Schrödinger equation of the Kepler potential with an extra potential barrier†.

The q-oscillator energy E_q is dependent on ω and q. From the identification of the coupling constant

$$-\beta^2 = E_q \tag{35}$$

we can solve ω in terms of β and q, as $\omega(\beta, q)$. Hence, in terms of the solutions of the q-Schrödinger equation for q-oscillator (26) we can obtain the solutions of

$$\left[\frac{-1}{2\mu}D_{q^2}^2(x) + \frac{\beta^2/2(1+q)^2}{x} + \frac{(2q^2-2q-3)/8\mu q^2(1+q)^2}{x^2}\right]\varphi_q(x) = E_{\mathrm{K}}\varphi_q(x) \tag{36}$$

where

$$E_{\mathrm{K}} = \frac{q\mu[\omega(\beta,q)]^2}{2(1+q)^2} \tag{37}$$

is the q-deformed analogue of the energy spectrum of the Kepler problem.

5. Energy spectrum and ground-state wavefunctions

A general solution of the q-Schrödinger equation of the q-deformed oscillator (26) is not known. We fix the energy spectrum to be of the conventional form [2]:

$$E_{qn} = [\omega(2n+1)]_q \equiv \frac{1-q^{\omega(2n+1)}}{1-q}. \tag{38}$$

Substituting the above energy spectrum into (35) we obtain

$$\omega_{qn}(\beta, q) = \frac{1-[(1-q)(1+\beta^2)]^{\ln q/2(2n+1)}}{1-q}. \tag{39}$$

The energy spectrum (37) of the q-Kepler problem then becomes

$$E_{\mathrm{K}n} = \frac{q\mu}{2(1-q^2)^2}\left\{1-[(1-q)(1+\beta^2)]^{\ln q/2(2n+1)}\right\}^2. \tag{40}$$

We would now like to build the ground-state wavefunction of the q-oscillator. With this aim we introduce

$$e_q(z^2) = 1 + \sum_{n=1}^{\infty}\left(\prod_{k=1}^{n}\frac{2(1-q)}{1-q^{2k}}\right)z^{2n} \tag{41}$$

which is defined to satisfy

$$D_q(z)e_q(z^2) = 2ze_q(z^2).$$

† For a study of the q-deformed H-atom in a manner unrelated to the q-oscillator, see [11].

Hence, the equation for the ground state of the q-oscillator

$$\left(-\frac{1}{2\mu}D_q^2(u)+\frac{q}{2}\mu[\omega]_q^2u^2\right)\psi_q^0(u)=\tfrac{1}{2}[\omega]_q\psi_q^0(u) \tag{42}$$

possesses the solution

$$\psi_q^0(u)=e_q\left(-\frac{1}{2\mu}\frac{1-q^{\omega}}{1-q}u^2\right). \tag{43}$$

By introducing the above definition into (32) and using (33) we obtain the ground-state wavefunction of the q-Kepler problem:

$$\varphi_q^0(x)=x^{-\alpha}\psi_q^0(\sqrt{x})=x^{-\alpha}e_q\left(-\frac{1}{2\mu}\frac{1-q^{\omega}}{1-q}x\right) \tag{44}$$

which corresponds to the energy E_{K0}.

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